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# Edge-disjoint double rays in infinite graphs: a Halin type result

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## Abstract

We show that any graph that contains  $k$  edge-disjoint double rays for any  $k \in \mathbb{N}$  contains also infinitely many edge-disjoint double rays. This was conjectured by Andreae in 1981.

## 1 Introduction

We say a graph  $G$  has *arbitrarily many vertex-disjoint*  $H$  if for every  $k \in \mathbb{N}$  there is a family of  $k$  vertex-disjoint subgraphs of  $G$  each of which is isomorphic to  $H$ . Halin's Theorem says that every graph that has arbitrarily many vertex-disjoint rays, also has infinitely many vertex-disjoint rays [5]. In 1970 he extended this result to vertex-disjoint double rays [6]. Jung proved a strengthening of Halin's Theorem where the initial vertices of the rays are constrained to a certain vertex set [7].

We look at the same questions with 'edge-disjoint' replacing 'vertex-disjoint'. Consider first the statement corresponding to Halin's Theorem. It suffices to prove this statement in locally finite graphs, as each graph with arbitrarily many edge-disjoint rays contains a locally finite union of tails of these rays. But the statement for locally finite graphs follows from Halin's original Theorem applied to the line-graph.

This reduction to locally finite graphs does not work for Jung's Theorem or for Halin's statement about double rays. Andreae proved an analog of Jung's Theorem for edge-disjoint rays in 1981, and conjectured that a Halin-type Theorem would be true for edge-disjoint double rays [1]. Our aim in the current paper is to prove this conjecture.

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More precisely, we say a graph  $G$  has *arbitrarily many edge-disjoint  $H$*  if for every  $k \in \mathbb{N}$  there is a family of  $k$  edge-disjoint subgraphs of  $G$  each of which is isomorphic to  $H$ , and our main result is the following.

**Theorem 1.** *Any graph that has arbitrarily many edge-disjoint double rays has infinitely many edge-disjoint double rays.*

Even for locally finite graphs this theorem does not follow from Halin's analogous result for vertex-disjoint double rays applied to the line graph. For example a double ray in the line graph may correspond, in the original graph, to a configuration as in Figure 1.

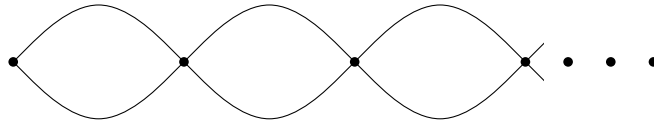


Figure 1: A graph that does not include a double ray but whose line graph does.

A related notion is that of ubiquity. A graph  $H$  is *ubiquitous* with respect to a graph relation  $\leq$  if  $nH \leq G$  for all  $n \in \mathbb{N}$  implies  $\aleph_0 H \leq G$ , where  $nH$  denotes the disjoint union of  $n$  copies of  $H$ . For example, Halin's Theorem says that rays are ubiquitous with respect to the subgraph relation. It is known that not every graph is ubiquitous with respect to the minor relation [2], nor is every locally finite graph ubiquitous with respect to the subgraph relation [8, 9], or even the topological minor relation [2, 3]. However, Andreae has conjectured that every locally finite graph is ubiquitous with respect to the minor relation [2]. For more details see [3]. In Section 6 (the outlook) we introduce a notion closely related to ubiquity.

The proof is organised as follows. In Section 3 we explain how to deal with the cases that the graph has infinitely many ends, or an end with infinite vertex-degree. In Section 4 we consider the 'two ended' case: That in which there are two ends  $\omega$  and  $\omega'$  both of finite vertex-degree, and arbitrarily many edge-disjoint double rays from  $\omega$  to  $\omega'$ .

The only remaining case is the 'one ended' case: That in which there is a single end  $\omega$  of finite vertex-degree and arbitrarily many edge-disjoint double rays from  $\omega$  to  $\omega$ . One central idea in the proof of this case is to consider 2-rays instead of double rays. Here a 2-ray is a pair of vertex-disjoint rays. For example, from each double ray one obtains a 2-ray by removing a finite path. The remainder of the proof is subdivided into two parts: In Subsection 5.3 we show that if there are arbitrarily many edge-disjoint 2-rays into  $\omega$ , then there are infinitely many such 2-rays. In Subsection 5.2 we show that if there are infinitely

many edge-disjoint 2-rays into  $\omega$ , then there are infinitely many edge-disjoint double rays from  $\omega$  to  $\omega$ .

We finish by discussing the outlook and mentioning some open problems.

## 2 Preliminaries

All our basic notation for graphs is taken from [4]. In particular, two rays in a graph are equivalent if no finite set separates them. The equivalence classes of this relation are called the *ends* of  $G$ . We say that a ray in an end  $\omega$  *converges* to  $\omega$ . A double ray *converges* to all the ends of which it includes a ray.

### 2.1 The structure of a thin end

It follows from Halin's Theorem that if there are arbitrarily many vertex-disjoint rays in an end of  $G$ , then there are infinitely many such rays. This fact motivated the central definition of the *vertex-degree* of an end  $\omega$ : the maximal cardinality of a set of vertex-disjoint rays in  $\omega$ .

An end is *thin* if its vertex-degree is finite, and otherwise it is *thick*. A pair  $(A, B)$  of edge-disjoint subgraphs of  $G$  is a *separation* of  $G$  if  $A \cup B = G$ . The number of vertices of  $A \cap B$  is called the *order* of the separation.

**Definition 2.** Let  $G$  be a locally finite graph and  $\omega$  a thin end of  $G$ . A countable infinite sequence  $((A_i, B_i))_{i \in \mathbb{N}}$  of separations of  $G$  *captures*  $\omega$  if for all  $i \in \mathbb{N}$

- $A_i \cap B_{i+1} = \emptyset$ ,
- $A_{i+1} \cap B_i$  is connected,
- $\bigcup_{i \in \mathbb{N}} A_i = G$ ,
- the order of  $(A_i, B_i)$  is the vertex-degree of  $\omega$ , and
- each  $B_i$  contains a ray from  $\omega$ .

**Lemma 3.** *Let  $G$  be a locally finite graph with a thin end  $\omega$ . Then there is a sequence that captures  $\omega$ .*

*Proof.* Without loss of generality  $G$  is connected, and so is countable. Let  $v_1, v_2, \dots$  be an enumeration of the vertices of  $G$ . Let  $k$  be the vertex-degree of  $\omega$ . Let  $\mathcal{R} = \{R_1, \dots, R_k\}$  be a set of vertex-disjoint rays in  $\omega$  and let  $S$  be the set of their start vertices. We pick a sequence  $((A_i, B_i))_{i \in \mathbb{N}}$  of separations and a sequence  $(T_i)$  of connected subgraphs recursively as follows. We pick  $(A_i, B_i)$  such that  $S$  is included in  $A_i$ , such that there is a ray from  $\omega$  included in  $B_i$ , and such that  $B_i$  does not meet  $\bigcup_{j < i} T_j$  or  $\{v_j \mid j \leq i\}$ : subject to this we minimise the size of the set  $X_i$  of vertices in  $A_i \cap B_i$ . Because of this minimization  $B_i$  is connected and  $X_i$  is finite. We take  $T_i$  to be a finite connected subgraph of  $B_i$  including  $X_i$ . Note that any ray that meets all of the  $B_i$  must be in  $\omega$ .

By Menger's Theorem [4] we get for each  $i \in \mathbb{N}$  a set  $\mathcal{P}_i$  of vertex-disjoint paths from  $X_i$  to  $X_{i+1}$  of size  $|X_i|$ . From these, for each  $i$  we get a set of  $|X_i|$  vertex-disjoint rays in  $\omega$ . Thus the size of  $X_i$  is at most  $k$ . On the other hand it is at least  $k$  as each ray  $R_j$  meets each set  $X_i$ .

Assume for contradiction that there is a vertex  $v \in A_i \cap B_{i+1}$ . Let  $R$  be a ray from  $v$  to  $\omega$  inside  $B_{i+1}$ . Then  $R$  must meet  $X_i$ , contradicting the definition of  $B_{i+1}$ . Thus  $A_i \cap B_{i+1}$  is empty.

Observe that  $\bigcup \mathcal{P}_i \cup T_i$  is a connected subgraph of  $A_{i+1} \cap B_i$  containing all vertices of  $X_i$  and  $X_{i+1}$ . For any vertex  $v \in A_{i+1} \cap B_i$  there is a  $v$ - $X_{i+1}$  path  $P$  in  $B_i$ .  $P$  meets  $B_{i+1}$  only in  $X_{i+1}$ . So  $P$  is included in  $A_{i+1} \cap B_i$ . Thus  $A_{i+1} \cap B_i$  is connected. The remaining conditions are clear.  $\square$

**Remark 4.** *Every infinite subsequence of a sequence capturing  $\omega$  also captures  $\omega$ .*  $\square$

The following is obvious:

**Remark 5.** *Let  $G$  be a graph and  $v, w \in V(G)$ . If  $G$  contains arbitrarily many edge-disjoint  $v$ - $w$  paths, then it contains infinitely many edge-disjoint  $v$ - $w$  paths.*  $\square$

We will need the following special case of the theorem of Andreae mentioned in the Introduction.

**Theorem 6** (Andreae [1]). *Let  $G$  be a graph and  $v \in V(G)$ . If there are arbitrarily many edge-disjoint rays all starting at  $v$ , then there are infinitely many edge-disjoint rays all starting at  $v$ .*

### 3 Known cases

Many special cases of Theorem 1 are already known or easy to prove. For example Halin showed the following.

**Theorem 7** (Halin). *Let  $G$  be a graph and  $\omega$  an end of  $G$ . If  $\omega$  contains arbitrarily many vertex-disjoint rays, then  $G$  has a half-grid as a minor.*

**Corollary 8.** *Any graph with an end of infinite vertex-degree has infinitely many edge-disjoint double rays.*  $\square$

Another simple case is the case where the graph has infinitely many ends.

**Lemma 9.** *A tree with infinitely many ends contains infinitely many edge-disjoint double rays.*

*Proof.* It suffices to show that every tree  $T$  with infinitely many ends contains a double ray such that removing its edges leaves a component containing infinitely many ends, since then one can pick those double rays recursively.

There is a vertex  $v \in V(T)$  such that  $T - v$  has at least 3 components  $C_1, C_2, C_3$  that each have at least one end, as  $T$  contains more than 2 ends. Let

$e_i$  be the edge  $vw_i$  with  $w_i \in C_i$  for  $i \in \{1, 2, 3\}$ . The graph  $T \setminus \{e_1, e_2, e_3\}$  has precisely 4 components ( $C_1, C_2, C_3$  and the one containing  $v$ ), one of which,  $D$  say, has infinitely many ends. By symmetry we may assume that  $D$  is neither  $C_1$  nor  $C_2$ . There is a double ray  $R$  all whose edges are contained in  $C_1 \cup C_2 \cup \{e_1, e_2\}$ . Removing the edges of  $R$  leaves the component  $D$ , which has infinitely many ends.  $\square$

**Corollary 10.** *Any connected graph with infinitely many ends has infinitely many edge-disjoint double rays.*  $\square$

## 4 The ‘two ended’ case

Using the results of Section 3 it is enough to show that any graph with only finitely many ends, each of which is thin, has infinitely many edge-disjoint double rays as soon as it has arbitrarily many edge-disjoint double rays. Any double ray in such a graph has to join a pair of ends (not necessarily distinct), and there are only finitely many such pairs. So if there are arbitrarily many edge-disjoint double rays, then there is a pair of ends such that there are arbitrarily many edge-disjoint double rays joining those two ends. In this section we deal with the case where these two ends are different, and in Section 5 we deal with the case that they are the same. We start with two preparatory lemmas.

**Lemma 11.** *Let  $G$  be a graph with a thin end  $\omega$ , and let  $\mathcal{R} \subseteq \omega$  be an infinite set. Then there is an infinite subset of  $\mathcal{R}$  such that any two of its members intersect in infinitely many vertices.*

*Proof.* We define an auxiliary graph  $H$  with  $V(H) = \mathcal{R}$  and an edge between two rays if and only if they intersect in infinitely many vertices. By Ramsey’s Theorem either  $H$  contains an infinite clique or an infinite independent set of vertices. Let us show that there cannot be an infinite independent set in  $H$ . Let  $k$  be the vertex-degree of  $\omega$ : we shall show that  $H$  does not have an independent set of size  $k + 1$ . Suppose for a contradiction that  $X \subseteq \mathcal{R}$  is a set of  $k + 1$  rays that is independent in  $H$ . Since any two rays in  $X$  meet in only finitely many vertices, each ray in  $X$  contains a tail that is disjoint to all the other rays in  $X$ . The set of these  $k + 1$  vertex-disjoint tails witnesses that  $\omega$  has vertex-degree at least  $k + 1$ , a contradiction. Thus there is an infinite clique  $K \subseteq H$ , which is the desired infinite subset.  $\square$

**Lemma 12.** *Let  $G$  be a graph consisting of the union of a set  $\mathcal{R}$  of infinitely many edge-disjoint rays of which any pair intersect in infinitely many vertices. Let  $X \subseteq V(G)$  be an infinite set of vertices, then there are infinitely many edge-disjoint rays in  $G$  all starting in different vertices of  $X$ .*

*Proof.* If there are infinitely many rays in  $\mathcal{R}$  each of which contains a different vertex from  $X$ , then suitable tails of these rays give the desired rays. Otherwise there is a ray  $R \in \mathcal{R}$  meeting  $X$  infinitely often. In this case, we choose the desired rays recursively such that each contains a tail from some ray in  $\mathcal{R} - R$ .

Having chosen finitely many such rays, we can always pick another: we start at some point in  $X$  on  $R$  which is beyond all the (finitely many) edges on  $R$  used so far. We follow  $R$  until we reach a vertex of some ray  $R'$  in  $\mathcal{R} - R$  whose tail has not been used yet, then we follow  $R'$ .  $\square$

**Lemma 13.** *Let  $G$  be a graph with only finitely many ends, all of which are thin. Let  $\omega_1, \omega_2$  be distinct ends of  $G$ . If  $G$  contains arbitrarily many edge-disjoint double rays each of which converges to both  $\omega_1$  and  $\omega_2$ , then  $G$  contains infinitely many edge-disjoint double rays each of which converges to both  $\omega_1$  and  $\omega_2$ .*

*Proof.* For each pair of ends, there is a finite set separating them. The finite union of these finite sets is a finite set  $S \subseteq V(G)$  separating any two ends of  $G$ . For  $i = 1, 2$  let  $C_i$  be the component of  $G - S$  containing  $\omega_i$ .

There are arbitrarily many edge-disjoint double rays from  $\omega_1$  to  $\omega_2$  that have a common last vertex  $v_1$  in  $S$  before staying in  $C_1$  and also a common last vertex  $v_2$  in  $S$  before staying in  $C_2$ . Note that  $v_1$  may be equal to  $v_2$ . There are arbitrarily many edge-disjoint rays in  $C_1 + v_1$  all starting in  $v_1$ . By Theorem 6 there is a countable infinite set  $\mathcal{R}_1 = \{R_1^i \mid i \in \mathbb{N}\}$  of edge-disjoint rays each included in  $C_1 + v_1$  and starting in  $v_1$ . By replacing  $\mathcal{R}_1$  with an infinite subset of itself, if necessary, we may assume by Lemma 11 that any two members of  $\mathcal{R}_1$  intersect in infinitely many vertices. Similarly, there is a countable infinite set  $\mathcal{R}_2 = \{R_2^i \mid i \in \mathbb{N}\}$  of edge-disjoint rays each included in  $C_2 + v_2$  and starting in  $v_2$  such that any two members of  $\mathcal{R}_2$  intersect in infinitely many vertices.

Let us subdivide all edges in  $\bigcup \mathcal{R}_1$  and call the set of subdivision vertices  $X_1$ . Similarly, we subdivide all edges in  $\bigcup \mathcal{R}_2$  and call the set of subdivision vertices  $X_2$ . Below we shall find double rays in the subdivided graph, which immediately give rise to the desired double rays in  $G$ .

Suppose for a contradiction that there is a finite set  $F$  of edges separating  $X_1$  from  $X_2$ . Then  $v_i$  has to be on the same side of that separation as  $X_i$  as there are infinitely many  $v_i - X_i$  edges. So  $F$  separates  $v_1$  from  $v_2$ , which contradicts the fact that there are arbitrarily many edge-disjoint double rays containing both  $v_1$  and  $v_2$ . By Remark 5 there is a set  $\mathcal{P}$  of infinitely many edge-disjoint  $X_1 - X_2$  paths. As all vertices in  $X_1$  and  $X_2$  have degree 2, and by taking an infinite subset if necessary, we may assume that each end-vertex of a path in  $\mathcal{P}$  lies on no other path in  $\mathcal{P}$ .

By Lemma 12 there is an infinite set  $Y_1$  of start-vertices of paths in  $\mathcal{P}$  together with an infinite set  $\mathcal{R}'_1$  of edge-disjoint rays with distinct start-vertices whose set of start-vertices is precisely  $Y_1$ . Moreover, we can ensure that each ray in  $\mathcal{R}'_1$  is included in  $\bigcup \mathcal{R}_1$ . Let  $Y_2$  be the set of end-vertices in  $X_2$  of those paths in  $\mathcal{P}$  that start in  $Y_1$ . Applying Lemma 12 again, we obtain an infinite set  $Z_2 \subseteq Y_2$  together with an infinite set  $\mathcal{R}'_2$  of edge-disjoint rays included in  $\bigcup \mathcal{R}_2$  with distinct start-vertices whose set of start-vertices is precisely  $Z_2$ .

For each path  $P$  in  $\mathcal{P}$  ending in  $Z_2$ , there is a double ray in the union of  $P$  and the two rays from  $\mathcal{R}'_1$  and  $\mathcal{R}'_2$  that  $P$  meets in its end-vertices. By construction, all these infinitely many double rays are edge-disjoint. Each of

those double rays converges to both  $\omega_1$  and  $\omega_2$ , since each  $\omega_i$  is the only end in  $C_i$ .  $\square$

**Remark 14.** *Instead of subdividing edges we also could have worked in the line graph of  $G$ . Indeed, there are infinitely many vertex-disjoint paths in the line graph from  $\bigcup \mathcal{R}_1$  to  $\bigcup \mathcal{R}_2$ .*

## 5 The ‘one ended’ case

We are now going to look at graphs  $G$  that contain a thin end  $\omega$  such that there are arbitrarily many edge-disjoint double rays converging only to the end  $\omega$ . The aim of this section is to prove the following lemma, and to deduce Theorem 1.

**Lemma 15.** *Let  $G$  be a countable graph and let  $\omega$  be a thin end of  $G$ . Assume there are arbitrarily many edge-disjoint double rays all of whose rays converge to  $\omega$ . Then  $G$  has infinitely many edge-disjoint double rays.*

We promise that the assumption of countability will not cause problems later.

### 5.1 Reduction to the locally finite case

A key notion for this section is that of a 2-ray. A *2-ray* is a pair of vertex-disjoint rays. For example, from each double ray one obtains a 2-ray by removing a finite path.

In order to deduce that  $G$  has infinitely many edge-disjoint double rays, we will only need that  $G$  has arbitrarily many edge-disjoint 2-rays. In this subsection, we illustrate one advantage of 2-rays, namely that we may reduce to the case where  $G$  is locally finite.

**Lemma 16.** *Let  $G$  be a countable graph with a thin end  $\omega$ . Assume there is a countable infinite set  $\mathcal{R}$  of rays all of which converge to  $\omega$ .*

*Then there is a locally finite subgraph  $H$  of  $G$  with a single end which is thin such that the graph  $H$  includes a tail of any  $R \in \mathcal{R}$ .*

*Proof.* Let  $(R_i \mid i \in \mathbb{N})$  be an enumeration of  $\mathcal{R}$ . Let  $(v_i \mid i \in \mathbb{N})$  be an enumeration of the vertices of  $G$ . Let  $U_i$  be the unique component of  $G \setminus \{v_1, \dots, v_i\}$  including a tail of each ray in  $\omega$ .

For  $i \in \mathbb{N}$ , we pick a tail  $R'_i$  of  $R_i$  in  $U_i$ . Let  $H_1 = \bigcup_{i \in \mathbb{N}} R'_i$ . Making use of  $H_1$ , we shall construct the desired subgraph  $H$ . Before that, we shall collect some properties of  $H_1$ .

As every vertex of  $G$  lies in only finitely many of the  $U_i$ , the graph  $H_1$  is locally finite. Each ray in  $H_1$  converges to  $\omega$  in  $G$  since  $H_1 \setminus U_i$  is finite for every  $i \in \mathbb{N}$ . Let  $\Psi$  be the set of ends of  $H_1$ . Since  $\omega$  is thin,  $\Psi$  has to be finite:  $\Psi = \{\omega_1, \dots, \omega_n\}$ . For each  $i \leq n$ , we pick a ray  $S_i \subseteq H_1$  converging to  $\omega_i$ .

Now we are in a position to construct  $H$ . For any  $i > 1$ , the rays  $S_1$  and  $S_i$  are joined by an infinite set  $\mathcal{P}_i$  of vertex-disjoint paths in  $G$ . We obtain  $H$  from



$H_1$  by adding all paths in the sets  $\mathcal{P}_i$ . Since  $H_1$  is locally finite,  $H$  is locally finite.

It remains to show that every ray  $R$  in  $H$  is equivalent to  $S_1$ . If  $R$  contains infinitely many edges from the  $\mathcal{P}_i$ , then there is a single  $\mathcal{P}_i$  which  $R$  meets infinitely, and thus  $R$  is equivalent to  $S_1$ . Thus we may assume that a tail of  $R$  is a ray in  $H_1$ . So it converges to some  $\omega_i \in \Psi$ . Since  $S_i$  and  $S_1$  are equivalent,  $R$  and  $S_1$  are equivalent, which completes the proof.  $\square$

**Corollary 17.** *Let  $G$  be a countable graph with a thin end  $\omega$  and arbitrarily many edge-disjoint 2-rays of which all the constituent rays converge to  $\omega$ . Then there is a locally finite subgraph  $H$  of  $G$  with a single end, which is thin, such that  $H$  has arbitrarily many edge-disjoint 2-rays.*

*Proof.* By Lemma 16 there is a locally finite graph  $H \subseteq G$  with a single end such that a tail of each of the constituent rays of the arbitrarily many 2-rays is included in  $H$ .  $\square$

## 5.2 Double rays versus 2-rays

A connected subgraph of a graph  $G$  including a vertex set  $S \subseteq V(G)$  is a *connector* of  $S$  in  $G$ .

**Lemma 18.** *Let  $G$  be a connected graph and  $S$  a finite set of vertices of  $G$ . Let  $\mathcal{H}$  be a set of edge-disjoint subgraphs  $H$  of  $G$  such that each connected component of  $H$  meets  $S$ . Then there is a finite connector  $T$  of  $S$ , such that at most  $2|S| - 2$  graphs from  $\mathcal{H}$  contain edges of  $T$ .*

*Proof.* By replacing  $\mathcal{H}$  with the set of connected components of graphs in  $\mathcal{H}$ , if necessary, we may assume that each member of  $\mathcal{H}$  is connected. We construct graphs  $T_i$  recursively for  $0 \leq i < |S|$  such that each  $T_i$  is finite and has at most  $|S| - i$  components, at most  $2i$  graphs from  $\mathcal{H}$  contain edges of  $T_i$ , and each component of  $T_i$  meets  $S$ . Let  $T_0 = (S, \emptyset)$  be the graph with vertex set  $S$  and no edges. Assume that  $T_i$  has been defined.

If  $T_i$  is connected let  $T_{i+1} = T_i$ . For a component  $C$  of  $T_i$ , let  $C'$  be the graph obtained from  $C$  by adding all graphs from  $\mathcal{H}$  that meet  $C$ .

As  $G$  is connected, there is a path  $P$  (possibly trivial) in  $G$  joining two of these subgraphs  $C'_1$  and  $C'_2$  say. And by taking the length of  $P$  minimal, we may assume that  $P$  does not contain any edge from any  $H \in \mathcal{H}$ . Then we can extend  $P$  to a  $C_1$ – $C_2$  path  $Q$  by adding edges from at most two subgraphs from  $\mathcal{H}$  — one included in  $C'_1$  and the other in  $C'_2$ . We obtain  $T_{i+1}$  from  $T_i$  by adding  $Q$ .

$T = T_{|S|-1}$  has at most one component and thus is connected. And at most  $2|S| - 2$  many graphs from  $\mathcal{H}$  contain edges of  $T$ . Thus  $T$  is as desired.  $\square$

Let  $d, d'$  be 2-rays.  $d$  is a *tail* of  $d'$  if each ray of  $d$  is a tail of a ray of  $d'$ . A set  $D'$  is a *tailor* of a set  $D$  of 2-rays if each element of  $D'$  is a tail of some element of  $D$  but no 2-ray in  $D$  includes more than one 2-ray in  $D'$ .

**Lemma 19.** *Let  $G$  be a locally finite graph with a single end  $\omega$ , which is thin. Assume that  $G$  contains an infinite set  $D = \{d_1, d_2, \dots\}$  of edge-disjoint 2-rays.*

*Then  $G$  contains an infinite tailer  $D'$  of  $D$  and a sequence  $((A_i, B_i))_{i \in \mathbb{N}}$  capturing  $\omega$  (see Definition 2) such that there is a family of vertex-disjoint connectors  $T_i$  of  $A_i \cap B_i$  contained in  $A_{i+1} \cap B_i$ , each of which is edge-disjoint from each member of  $D'$ .*

*Proof.* Let  $k$  be the vertex-degree of  $\omega$ . By Lemma 3 there is a sequence  $((A'_i, B'_i))_{i \in \mathbb{N}}$  capturing  $\omega$ . By replacing each 2-ray in  $D$  with a tail of itself if necessary, we may assume that for all  $(r, s) \in D$  and  $i \in \mathbb{N}$  either both  $r$  and  $s$  meet  $A'_i$  or none meets  $A'_i$ . By Lemma 18 there is a finite connector  $T'_i$  of  $A'_i \cap B'_i$  in the connected graph  $B'_i$  which meets in an edge at most  $2k - 2$  of the 2-rays of  $D$  that have a vertex in  $A'_i$ .

Thus, there are at most  $2k - 2$  2-rays in  $D$  that meet all but finitely many of the  $T'_i$  in an edge. By throwing away these finitely many 2-rays in  $D$  we may assume that each 2-ray in  $D$  is edge-disjoint from infinitely many of the  $T'_i$ . So we can recursively build a sequence  $N_1, N_2, \dots$  of infinite sets of natural numbers such that  $N_i \supseteq N_{i+1}$ , the first  $i$  elements of  $N_i$  are all contained in  $N_{i+1}$ , and  $d_i$  only meets finitely many of the  $T'_j$  with  $j \in N_i$  in an edge. Then  $N = \bigcap_{i \in \mathbb{N}} N_i$  is infinite and has the property that each  $d_i$  only meets finitely many of the  $T'_j$  with  $j \in N$  in an edge. Thus there is an infinite tailer  $D'$  of  $D$  such that no 2-ray from  $D'$  meets any  $T'_j$  for  $j \in N$  in an edge.

We recursively define a sequence  $n_1, n_2, \dots$  of natural numbers by taking  $n_i \in N$  sufficiently large that  $B'_{n_i}$  does not meet  $T'_{n_j}$  for any  $j < i$ . Taking  $(A_i, B_i) = (A'_{n_i}, B'_{n_i})$  and  $T_i = T'_{n_i}$  gives the desired sequences.  $\square$

**Lemma 20.** *If a locally finite graph  $G$  with a single end  $\omega$  which is thin contains infinitely many edge-disjoint 2-rays, then  $G$  contains infinitely many edge-disjoint double rays.*

*Proof.* Applying Lemma 19 we get an infinite set  $D$  of edge-disjoint 2-rays, a sequence  $((A_i, B_i))_{i \in \mathbb{N}}$  capturing  $\omega$ , and connectors  $T_i$  of  $A_i \cap B_i$  for each  $i \in \mathbb{N}$  such that the  $T_i$  are vertex-disjoint from each other and edge-disjoint from all members of  $D$ .

We shall construct the desired set of infinitely many edge-disjoint double rays as a nested union of sets  $D_i$ . We construct the  $D_i$  recursively. Assume that a set  $D_i$  of  $i$  edge-disjoint double rays has been defined such that each of its members is included in the union of a single 2-ray from  $D$  and one connector  $T_j$ . Let  $d_{i+1} \in D$  be a 2-ray distinct from the finitely many 2-rays used so far. Let  $C_{i+1}$  be one of the infinitely many connectors that is different from all the finitely many connectors used so far and that meets both rays of  $d_{i+1}$ . Clearly,  $d_{i+1} \cup C_{i+1}$  includes a double ray  $R_{i+1}$ . Let  $D_{i+1} = D_i \cup \{R_{i+1}\}$ . The union  $\bigcup_{i \in \mathbb{N}} D_i$  is an infinite set of edge-disjoint double rays as desired.  $\square$

### 5.3 Shapes and allowed shapes

Let  $G$  be a graph and  $(A, B)$  a separation of  $G$ . A *shape* for  $(A, B)$  is a word  $v_1x_1v_2x_2\ldots x_{n-1}v_n$  with  $v_i \in A \cap B$  and  $x_i \in \{l, r\}$  such that no vertex appears twice. We call the  $v_i$  the *vertices* of the shape. Every ray  $R$  induces a shape  $\sigma = \sigma_R(A, B)$  on every separation  $(A, B)$  of finite order in the following way: Let  $<_R$  be the *natural order* on  $V(R)$  induced by the ray, where  $v <_R w$  if  $w$  lies in the unique infinite component of  $R - v$ . The vertices of  $\sigma$  are those vertices of  $R$  that lie in  $A \cap B$  and they appear in  $\sigma$  in the order given by  $<_R$ . For  $v_i, v_{i+1}$  the path  $v_i R v_{i+1}$  has edges only in  $A$  or only in  $B$  but not in both. In the first case we put  $l$  between  $v_i$  and  $v_{i+1}$  and in the second case we put  $r$  between  $v_i$  and  $v_{i+1}$ .

Let  $(A_1, B_1), (A_2, B_2)$  be separations with  $A_1 \cap B_2 = \emptyset$  and thus also  $A_1 \subseteq A_2$  and  $B_2 \subseteq B_1$ . Let  $\sigma_i$  be a nonempty shape for  $(A_i, B_i)$ . The word  $\tau = v_1x_1v_2\ldots x_{n-1}v_n$  is an *allowed shape linking  $\sigma_1$  to  $\sigma_2$*  with *vertices  $v_1 \ldots v_n$*  if the following holds.

- $v$  is a vertex of  $\tau$  if and only if it is a vertex of  $\sigma_1$  or  $\sigma_2$ ,
- if  $v$  appears before  $w$  in  $\sigma_i$ , then  $v$  appears before  $w$  in  $\tau$ ,
- $v_1$  is the initial vertex of  $\sigma_1$  and  $v_n$  is the terminal vertex of  $\sigma_2$ ,
- $x_i \in \{l, m, r\}$ ,
- the subword  $vlw$  appears in  $\tau$  if and only if it appears in  $\sigma_1$ ,
- the subword  $vrw$  appears in  $\tau$  if and only if it appears in  $\sigma_2$ ,
- $v_i \neq v_j$  for  $i \neq j$ .

Each ray  $R$  defines a word  $\tau = \tau_R[(A_1, B_1), (A_2, B_2)] = v_1x_1v_2\ldots x_{n-1}v_n$  with vertices  $v_i$  and  $x_i \in \{l, m, r\}$  as follows. The vertices of  $\tau$  are those vertices of  $R$  that lie in  $A_1 \cap B_1$  or  $A_2 \cap B_2$  and they appear in  $\tau$  in the order given by  $<_R$ . For  $v_i, v_{i+1}$  the path  $v_i R v_{i+1}$  has edges either only in  $A_1$ , only in  $A_2 \cap B_1$ , or only in  $B_2$ . In the first case we set  $x_i = l$  and  $\tau$  contains the subword  $v_i l v_{i+1}$ . In the second case we set  $x_i = m$  and  $\tau$  contains the subword  $v_i m v_{i+1}$ . In the third case we set  $x_i = r$  and  $\tau$  contains the subword  $v_i r v_{i+1}$ .

For a ray  $R$  to induce an allowed shape  $\tau_R[(A_1, B_1), (A_2, B_2)]$  we need at least that  $R$  starts in  $A_2$ . However, each ray in  $\omega$  has a tail such that whenever it meets an  $A_i$  it also starts in that  $A_i$ . Let us call such rays *lefty*. A 2-ray is *lefty* if both its rays are.

**Remark 21.** Let  $(A_1, B_1)$ , and  $(A_2, B_2)$  be two separations of finite order with  $A_1 \subseteq A_2$ , and  $B_2 \subseteq B_1$ . For every lefty ray  $R$  meeting  $A_1$ , the word  $\tau_R[(A_1, B_1), (A_2, B_2)]$  is an allowed shape linking  $\sigma_R(A_1, B_1)$  and  $\sigma_R(A_2, B_2)$ .  $\square$

From now on let us fix a locally finite graph  $G$  with a thin end  $\omega$  of vertex-degree  $k$ . And let  $((A_i, B_i))_{i \in \mathbb{N}}$  be a sequence capturing  $\omega$  such that each member has order  $k$ .

A *2-shape* for a separation  $(A, B)$  is a pair of shapes for  $(A, B)$ . Every 2-ray induces a 2-shape coordinatewise in the obvious way. Similarly, an *allowed 2-shape* is a pair of allowed shapes.

Clearly, there is a global constant  $c_1 \in \mathbb{N}$  depending only on  $k$  such that there are at most  $c_1$  distinct 2-shapes for each separation  $(A_i, B_i)$ . Similarly, there is a global constant  $c_2 \in \mathbb{N}$  depending only on  $k$  such that for all  $i, j \in \mathbb{N}$  there are at most  $c_2$  distinct allowed 2-shapes linking a 2-shape for  $(A_i, B_i)$  with a 2-shape for  $(A_j, B_j)$ .

For most of the remainder of this subsection we assume that for every  $i \in \mathbb{N}$  there is a set  $D_i$  consisting of at least  $c_1 \cdot c_2 \cdot i$  edge-disjoint 2-rays in  $G$ . Our aim will be to show that in these circumstances there must be infinitely many edge-disjoint 2-rays.

By taking a tailor if necessary, we may assume that every 2-ray in each  $D_i$  is lefty.

**Lemma 22.** *There is an infinite set  $J \subseteq \mathbb{N}$  and, for each  $i \in \mathbb{N}$ , a tailor  $D'_i$  of  $D_i$  of cardinality  $c_2 \cdot i$  such that for all  $i \in \mathbb{N}$  and  $j \in J$  all 2-rays in  $D'_i$  induce the same 2-shape  $\sigma[i, j]$  on  $(A_j, B_j)$ .*

*Proof.* We recursively build infinite sets  $J_i \subseteq \mathbb{N}$  and tailors  $D'_i$  of  $D_i$  such that for all  $k \leq i$  and  $j \in J_i$  all 2-rays in  $D'_k$  induce the same 2-shape on  $(A_j, B_j)$ . For all  $i \geq 1$ , we shall ensure that  $J_i$  is an infinite subset of  $J_{i-1}$  and that the  $i - 1$  smallest members of  $J_i$  and  $J_{i-1}$  are the same. We shall take  $J$  to be the intersection of all the  $J_i$ .

Let  $J_0 = \mathbb{N}$  and let  $D'_0$  be the empty set. Now, for some  $i \geq 1$ , assume that sets  $J_k$  and  $D'_k$  have been defined for all  $k < i$ . By replacing 2-rays in  $D_i$  by their tails, if necessary, we may assume that each 2-ray in  $D_i$  avoids  $A_\ell$ , where  $\ell$  is the  $(i - 1)$ st smallest value of  $J_{i-1}$ . As  $D_i$  contains  $c_1 \cdot c_2 \cdot i$  many 2-rays, for each  $j \in J_{i-1}$  there is a set  $S_j \subseteq D_i$  of size at least  $c_2 \cdot i$  such that each 2-ray in  $S_j$  induces the same 2-shape on  $(A_j, B_j)$ . As there are only finitely many possible choices for  $S_j$ , there is an infinite subset  $J_i$  of  $J_{i-1}$  on which  $S_j$  is constant. For  $D'_i$  we pick this value of  $S_j$ . Since each  $d \in D'_i$  induces the empty 2-shape on each  $(A_k, B_k)$  with  $k \leq \ell$  we may assume that the first  $i - 1$  elements of  $J_{i-1}$  are also included in  $J_i$ .

It is immediate that the set  $J = \bigcap_{i \in \mathbb{N}} J_i$  and the  $D'_i$  have the desired property.  $\square$

**Lemma 23.** *There are two strictly increasing sequences  $(n_i)_{i \in \mathbb{N}}$  and  $(j_i)_{i \in \mathbb{N}}$  with  $n_i \in \mathbb{N}$  and  $j_i \in J$  for all  $i \in \mathbb{N}$  such that  $\sigma[n_i, j_i] = \sigma[n_{i+1}, j_i]$  and  $\sigma[n_i, j_i]$  is not empty.*

*Proof.* Let  $H$  be the graph on  $\mathbb{N}$  with an edge  $vw \in E(H)$  if and only if there are infinitely many elements  $j \in J$  such that  $\sigma[v, j] = \sigma[w, j]$ .

As there are at most  $c_1$  distinct 2-shapes for any separator  $(A_i, B_i)$ , there is no independent set of size  $c_1 + 1$  in  $H$  and thus no infinite one. Thus, by Ramsey's theorem, there is an infinite clique in  $H$ . We may assume without loss of generality that  $H$  itself is a clique by moving to a subsequence of the  $D'_i$  if necessary. With this assumption we simply pick  $n_i = i$ .

Now we pick the  $j_i$  recursively. Assume that  $j_i$  has been chosen. As  $i$  and  $i + 1$  are adjacent in  $H$ , there are infinitely many indices  $\ell \in \mathbb{N}$  such that  $\sigma[i, \ell] = \sigma[i + 1, \ell]$ . In particular, there is such an  $\ell > j_i$  such that  $\sigma[i + 1, \ell]$  is not empty. We pick  $j_{i+1}$  to be one of those  $\ell$ .

Clearly,  $(j_i)_{i \in \mathbb{N}}$  is an increasing sequence and  $\sigma[i, j_i] = \sigma[i + 1, j_i]$  as well as  $\sigma[i, j_i]$  is non-empty for all  $i \in \mathbb{N}$ , which completes the proof.  $\square$

By moving to a subsequence of  $(D'_i)$  and  $((A_j, B_j))$ , if necessary, we may assume by Lemma 22 and Lemma 23 that for all  $i, j \in \mathbb{N}$  all  $d \in D'_i$  induce the same 2-shape  $\sigma[i, j]$  on  $(A_j, B_j)$ , and that  $\sigma[i, i] = \sigma[i + 1, i]$ , and that  $\sigma[i, i]$  is non-empty.

**Lemma 24.** *For all  $i \in \mathbb{N}$  there is  $D''_i \subseteq D'_i$  such that  $|D''_i| = i$ , and all  $d \in D''_i$  induce the same allowed 2-shape  $\tau[i]$  that links  $\sigma[i, i]$  and  $\sigma[i, i + 1]$ .*

*Proof.* Note that it is in this proof that we need all the 2-rays in  $D''_i$  to be lefty as they need to induce an allowed 2-shape that links  $\sigma[i, i]$  and  $\sigma[i, i + 1]$  as soon as they contain a vertex from  $A_i$ . As  $|D'_i| \geq i \cdot c_2$  and as there are at most  $c_2$  many distinct allowed 2-shapes that link  $\sigma[i, i]$  and  $\sigma[i, i + 1]$  there is  $D''_i \subseteq D'_i$  with  $|D''_i| = i$  such that all  $d \in D''_i$  induce the same allowed 2-shape.  $\square$

We enumerate the elements of  $D''_j$  as follows:  $d_1^j, d_2^j, \dots, d_j^j$ . Let  $(s_i^j, t_i^j)$  be a representation of  $d_i^j$ . Let  $S_i^j = s_i^j \cap A_{j+1} \cap B_j$ , and let  $\mathcal{S}_i = \bigcup_{j \geq i} S_i^j$ . Similarly, let  $T_i^j = t_i^j \cap A_{j+1} \cap B_j$ , and let  $\mathcal{T}_i = \bigcup_{j \geq i} T_i^j$ .

Clearly,  $\mathcal{S}_i$  and  $\mathcal{T}_i$  are vertex-disjoint and any two graphs in  $\bigcup_{i \in \mathbb{N}} \{\mathcal{S}_i, \mathcal{T}_i\}$  are edge-disjoint. We shall find a ray  $R_i$  in each of the  $\mathcal{S}_i$  and a ray  $R'_i$  in each of the  $\mathcal{T}_i$ . The infinitely many pairs  $(R_i, R'_i)$  will then be edge-disjoint 2-rays, as desired.

**Lemma 25.** *Each vertex  $v$  of  $\mathcal{S}_i$  has degree at most 2. If  $v$  has degree 1 it is contained in  $A_i \cap B_i$ .*

*Proof.* Clearly, each vertex  $v$  of  $\mathcal{S}_i$  that does not lie in any separator  $A_j \cap B_j$  has degree 2, as it is contained in precisely one  $S_i^j$ , and all the leaves of  $S_i^j$  lie in  $A_j \cap B_j$  and  $A_{j+1} \cap B_{j+1}$  as  $d_i^j$  is lefty. Indeed, in  $S_i^j$  it is an inner vertex of a path and thus has degree 2 in there. If  $v$  lies in  $A_i \cap B_i$  it has degree at most 2, as it is only a vertex of  $S_i^j$  for one value of  $j$ , namely  $j = i$ .

Hence, we may assume that  $v \in A_j \cap B_j$  for some  $j > i$ . Thus,  $\sigma[j, j]$  contains  $v$  and  $l : \sigma[j, j] : r$  contains precisely one of the four following subwords:

$$lvl, lvr, rvl, rvr$$

(Here we use the notation  $p : q$  to denote the concatenation of the word  $p$  with the word  $q$ .) In the first case  $\tau[j-1]$  contains  $mvm$  as a subword and  $\tau[j]$  has no  $m$  adjacent to  $v$ . Then  $S_i^{j-1}$  contains precisely 2 edges adjacent to  $v$  and  $S_i^j$  has no such edge. The fourth case is the first one with  $l$  and  $r$  and  $j$  and  $j-1$  interchanged.

In the second and third cases, each of  $\tau[j-1]$  and  $\tau[j]$  has precisely one  $m$  adjacent to  $v$ . So both  $S_i^{j-1}$  and  $S_i^j$  contain precisely 1 edge adjacent to  $v$ .

As  $v$  appears only as a vertex of  $S_i^\ell$  for  $\ell = j$  or  $\ell = j-1$ , the degree of  $v$  in  $S_i$  is 2.  $\square$

**Lemma 26.** *There are an odd number of vertices in  $S_i$  of degree 1.*

*Proof.* By Lemma 25 we have that each vertex of degree 1 lies in  $A_i \cap B_i$ . Let  $v$  be a vertex in  $A_i \cap B_i$ . Then,  $\sigma[i, i]$  contains  $v$  and  $l : \sigma[i, i] : r$  contains precisely one of the four following subwords:

$$lvl, lvr, rvl, rvr$$

In the first and fourth case  $v$  has even degree. It has degree 1 otherwise. As  $l : \sigma[i, i] : r$  starts with  $l$  and ends with  $r$ , the word  $lvr$  appear precisely once more than the word  $rvl$ . Indeed, between two occurrences of  $lvr$  there must be one of  $rvl$  and vice versa. Thus, there are an odd number of vertices with degree 1 in  $S_i$ .  $\square$

**Lemma 27.**  *$S_i$  includes a ray.*

*Proof.* By Lemma 25 every vertex of  $S_i$  has degree at most 2 and thus every component of  $S_i$  has at most two vertices of degree 1. By Lemma 26  $S_i$  has a component  $C$  that contains an odd number of vertices with degree 1. Thus  $C$  has precisely one vertex of degree 1 and all its other vertices have degree 2, thus  $C$  is a ray.  $\square$

**Corollary 28.**  *$G$  contains infinitely many edge-disjoint 2-rays.*

*Proof.* By symmetry, Lemma 27 is also true with  $\mathcal{T}_i$  in place of  $S_i$ . Thus  $S_i \cup \mathcal{T}_i$  includes a 2-ray  $X_i$ . The  $X_i$  are edge-disjoint by construction.  $\square$

Recall that Lemma 15 states that a countable graph with a thin end  $\omega$  and arbitrarily many edge-disjoint double rays all whose subrays converge to  $\omega$ , also has infinitely many edge-disjoint double rays. We are now in a position to prove this lemma.

*Proof of Lemma 15.* By Lemma 20 it suffices to show that  $G$  contains a subgraph  $H$  with a single end which is thin such that  $H$  has infinitely many edge-disjoint 2-rays. By Corollary 17,  $G$  has a subgraph  $H$  with a single end which is thin such that  $H$  has arbitrarily many edge-disjoint 2-rays. But then by the argument above  $H$  contains infinitely many edge-disjoint 2-rays, as required.  $\square$

With these tools at hand, the remaining proof of Theorem 1 is easy. Let us collect the results proved so far to show that each graph with arbitrarily many edge-disjoint double rays also has infinitely many edge-disjoint double rays.

*Proof of Theorem 1.* Let  $G$  be a graph that has a set  $D_i$  of  $i$  edge-disjoint double rays for each  $i \in \mathbb{N}$ . Clearly,  $G$  has infinitely many edge-disjoint double rays if its subgraph  $\bigcup_{i \in \mathbb{N}} D_i$  does, and thus we may assume without loss of generality that  $G = \bigcup_{i \in \mathbb{N}} D_i$ . In particular,  $G$  is countable.

By Corollary 10 we may assume that each connected component of  $G$  includes only finitely many ends. As each component includes a double ray we may assume that  $G$  has only finitely many components. Thus, there is one component containing arbitrarily many edge-disjoint double rays, and thus we may assume that  $G$  is connected.

By Corollary 8 we may assume that all ends of  $G$  are thin. Thus, as mentioned at the start of Section 4, there is a pair of ends  $(\omega, \omega')$  of  $G$  (not necessarily distinct) such that  $G$  contains arbitrarily many edge-disjoint double rays each of which converges precisely to  $\omega$  and  $\omega'$ . This completes the proof as, by Lemma 13  $G$  has infinitely many edge-disjoint double rays if  $\omega$  and  $\omega'$  are distinct and by Lemma 15  $G$  has infinitely many edge-disjoint double rays if  $\omega = \omega'$ .  $\square$

## 6 Outlook and open problems

We will say that a graph  $H$  is *edge-ubiquitous* if every graph having arbitrarily many edge-disjoint  $H$  also has infinitely many edge-disjoint  $H$ .

Thus Theorem 1 can be stated as follows: the double ray is edge-ubiquitous. Andreae's Theorem implies that the ray is edge-ubiquitous. And clearly, every finite graph is edge-ubiquitous.

We could ask which other graphs are edge-ubiquitous. It follows from our result that the 2-ray is edge-ubiquitous. Let  $G$  be a graph in which there are arbitrarily many edge-disjoint 2-rays. Let  $v * G$  be the graph obtained from  $G$  by adding a vertex  $v$  adjacent to all vertices of  $G$ . Then  $v * G$  has arbitrarily many edge-disjoint double rays, and thus infinitely many edge-disjoint double rays. Each of these double rays uses  $v$  at most once and thus includes a 2-ray of  $G$ .

The vertex-disjoint union of  $k$  rays is called a *k-ray*. The *k-ray* is edge-ubiquitous. This can be proved with an argument similar to that for Theorem 1: Let  $G$  be a graph with arbitrarily many edge-disjoint  $k$ -rays. The same argument as in Corollaries 10 and 8 shows that we may assume that  $G$  has only finitely many ends, each of which is thin. By removing a finite set of vertices if necessary we may assume that each component of  $G$  has at most one end, which is thin. Now we can find numbers  $k_C$  indexed by the components  $C$  of  $G$  and summing to  $k$  such that each component  $C$  has arbitrarily many edge-disjoint  $k_C$ -rays. Hence, we may assume that  $G$  has only a single end, which is thin. By Lemma 16 we may assume that  $G$  is locally finite.

In this case, we use an argument as in Subsection 5.3. It is necessary to use  $k$ -shapes instead of 2-shapes but other than that we can use the same combinatorial principle. If  $C_1$  and  $C_2$  are finite sets, a  $(C_1, C_2)$ -*shaping* is a pair  $(c_1, c_2)$  where  $c_1$  is a partial colouring of  $\mathbb{N}$  with colours from  $C_1$  which is defined at all but finitely many numbers and  $c_2$  is a colouring of  $\mathbb{N}^{(2)}$  with colours from  $C_2$  (in our argument above,  $C_1$  would be the set of all  $k$ -shapes and  $C_2$  would be the set of all allowed  $k$ -shapes for all pairs of  $k$ -shapes).

**Lemma 29.** *Let  $D_1, D_2, \dots$  be a sequence of sets of  $(C_1, C_2)$ -shapings where  $D_i$  has size  $i$ . Then there are strictly increasing sequences  $i_1, i_2, \dots$  and  $j_1, j_2, \dots$  and subsets  $S_n \subseteq D_{i_n}$  with  $|S_n| \geq n$  such that*

- *for any  $n \in \mathbb{N}$  all the values of  $c_1(j_n)$  for the shapings  $(c_1, c_2) \in S_{n-1} \cup S_n$  are equal (in particular, they are all defined).*
- *for any  $n \in \mathbb{N}$ , all the values of  $c_2(j_n, j_{n+1})$  for the shapings  $(c_1, c_2) \in S_n$  are equal.*

Lemma 29 can be proved by the same method with which we constructed the sets  $D''_i$  from the sets  $D_i$ . The advantage of Lemma 29 is that it can not only be applied to 2-rays but also to more complicated graphs like  $k$ -rays.

A *talon* is a tree with a single vertex of degree 3 where all the other vertices have degree 2. An argument as in Subsection 5.2 can be used to deduce that talons are edge-ubiquitous from the fact that 3-rays are. However, we do not know whether the graph in Figure 2 is edge-ubiquitous.

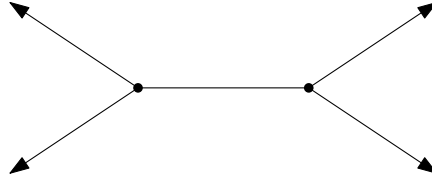


Figure 2: A graph obtained from 2 disjoint double rays, joined by a single edge. Is this graph edge-ubiquitous?

We finish with the following open problem.

**Problem 30.** *Is the directed analogue of Theorem 1 true? More precisely: Is it true that if a directed graph has arbitrarily many edge-disjoint directed double rays, then it has infinitely many edge-disjoint directed double rays?*

It should be noted that if true the directed analogue would be a common generalization of Theorem 1 and the fact that double rays are ubiquitous with respect to the subgraph relation.

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